Dawn · Midi · Yofuke International Seminar

Explicit Methods in Homotopy and Arithmetic

Program – Autumn, 2025

Organizers
B. Collas, RIMS Kyoto University, Japan
C. Rasmussen, Wesleyan University, USA

The *Dawn · Midi · Yofuke International Seminar* presents a mix of colloquium and research talks on focused themes in arithmetic homotopy geometry, a field which interacts with multiple areas, such as algebraic geometry, low-dimensional topology, number theory, or, at the homotopy-homology frontier, motivic theory and periods.

The intention is to both introduce classical topics to a wider audience, as well as report on recent progress. The first talk of each topic is given in a Colloquium style; the schedule has been chosen for connecting international communities, from US to Japan via Europe.

Homotopy Galois Action — September 2025

- 08 Sept. *Arithmetic of fundamental groups* .... C. Rasmussen, Wesleyan, US
- 09 Sept. *Galois actions on pro-p fundamental groups of once-punctured CM elliptic curves* .... S. Ishii, Keio, JP
- 22 Sept. *Arithmetic of elliptic curves related to Ihara's program* .... C. McLeman, Michigan-Flint, US
- 23 Sept. *Galois ℓ-adic polylogarithms* .... D. Shiraishi, Tokyo Univ., JP

Chabauty-Kim Theory — October 2025

- 07 Oct. *Arithmetic of nonabelian Chabauty* .... N. Dogra, King's College, UK
- 08 Oct. *An Oda–Tamagawa criterion for fundamental groupoids of hyperbolic curves* .... A. Betts, Cornell, US
- 13 Oct. *Explicit methods in Chabauty–Kim theory for rational points* .... S. Hashimoto, US
- 14 Oct. *Motivic aspects in genus zero* .... I. Dan-Cohen, Ben-Gurion, ISL

Talks are held in-person at Wesleyan University, and by Zoom at 08:00 EDT / 12:00 ETC / 21:00 JST.

For registration, room and Zoom access, and latest information, see: https://christopherrasmussen.github.io/dmyseminar/

This workshop is part of the LPP-RIMS-ENS Arithmetic Homotopy and Galois Theory RIMS-CNRS international project, and is supported by Wesleyan University and RIMS Kyoto University.

Arithmetic of fundamental groups

. . . . *Christopher* RASMUSSEN *:: Wesleyan University, USA*

This talk, aimed towards non-experts, will give a necessarily incomplete but self-contained survey of the theory of fundamental groups as they appear in arithmetic settings. We recall how the fundamental group generalizes both classic Galois theory and the theory of topological covering spaces. We explain the construction of the fundamental exact sequence attached to any reasonable scheme and how this sequence is used to construct natural Galois representations. We conclude with certain illustrations of anabelian philosophy, that certain geometric objects should be completely determined by their fundamental group together with the outer Galois action.

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Galois actions on pro-p fundamental groups of once-punctured CM elliptic curves

. . . . *Shun* ISHII *:: Keio University, Japan*

It has been observed since the late 1980s that the (outer) Galois action on the pro-p fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ encodes rich information about pro-p extensions with restricted ramification. A notable example is Sharifi's result, stating that the kernel of the associated (outer) Galois representation corresponds to the maximal pro-p extension of the p-th cyclotomic field unramified outside p if $p > 2$ is regular.

In this talk, we discuss our recent works toward developing similar results in the case of CM elliptic curves punctured at one point. We characterize the field corresponding to the kernel of the outer Galois representation under certain assumptions, and explain some recent results that further deepen the analogies between genus zero and genus one cases if time permits.

References.

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[Iha02] Y. IHARA, "Some arithmetic aspects of Galois actions in the pro-p fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$," in *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, ser. Proc. Sympos. Pure Math. Vol. 70, Amer. Math. Soc., Providence, RI, 2002, pp. 247–273, MR1935408.

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Arithmetic of elliptic curves related to Ihara's program

. . . . *Cam McLeman :: University of Michigan-Flint, USA*

For a number field K, consider the outer pro-ℓ Galois representation attached to the projective line over K minus three points. Ihara asked, for $K = \mathbb{Q}$, whether the field fixed by the kernel of this representation is precisely $\mathcal{T}$, the maximal pro-ℓ extension of $K(\mu_{\ell^\infty})$ unramified away from ℓ. The equality of the two fields has been affirmed in the case that ℓ is an odd regular prime by work of Brown and Sharifi, but many cases of Ihara's question remain open. To investigate further, it is valuable to find large subextensions of $\mathcal{T}$; we call an abelian variety A/K *heavenly* at ℓ precisely if $K(A[\ell^\infty]) \subseteq \mathcal{T}$.

It is known that for a fixed quadratic field K, the number of K-isomorphism classes of elliptic curves heavenly at some ℓ is finite, even running over all possible primes ℓ. We present a complementary result, that for a fixed prime $\ell \geq 7$, there are only finitely many isomorphism classes even as K runs over all quadratic fields. We explore the natural follow-up question – whether a finiteness result is available where both K and ℓ vary. We conjecture such a result, and present evidence in the form of striking similarity in the behavior for Frobenius traces attached to heavenly elliptic curves and to (certain) elliptic curves with complex multiplication.

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Galois ℓ-adic polylogarithms

. . . . *Densuke Shiraishi :: Tokyo University of Science, Japan*

The Galois ℓ-adic polylogarithm, introduced by Wojtkowiak, is a certain family of ℓ-adic numbers parameterized by elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Its primary purpose is to explicitly describe the Galois action on the pro-ℓ geometric étale fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with rational base points. The variant of polylogarithm along with its multiple version completely characterizes the pro-ℓ Galois action, enabling concrete numerical and quantitative analysis.

A noteworthy result is the explicit formula expressing the Galois ℓ-adic polylogarithm in terms of the generalized ℓ-adic Soulé character established by Nakamura and Wojtkowiak. This formula reveals deep connections between Galois ℓ-adic polylogarithms and various other arithmetic objects. Furthermore, the Galois ℓ-adic polylogarithm is the ℓ-adic étale counterpart to the complex polylogarithm. Analogously to the complex side, Galois ℓ-adic polylogarithms satisfy various functional equations.

In this talk, we will first introduce the definition and basic properties of Galois ℓ-adic polylogarithms. We will then review the foundational work of Nakamura and Wojtkowiak, followed by the speaker's results on these functional equations.

References.

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Topic Session :: Chabauty-Kim Theory

Arithmetic of nonabelian Chabauty

Netan Dogra :: King's College London, UK

The Chabauty–Coleman–Kim method aims to understand the Diophantine geometry of curves over number fields using a nonabelian generalisation of the Jacobian (or more precisely, its Selmer group). This is conjectured to give a formula for rational points on hyperbolic curves as the zeroes of certain p-adic interated integrals.

In this talk I will introduce, in varying degrees of detail, the main characters of the story: unipotent completions of fundamental groups, Bloch–Kato Selmer groups, Galois cohomology of unipotent groups, iterated integrals and nonabelian p-adic Hodge theory. I will also give some examples.

References.

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An Oda–Tamagawa criterion for fundamental groupoids of hyperbolic curves

Alexander Betts :: Cornell University, USA

A celebrated theorem of Takayuki Oda says that a curve over a p-adic field has good reduction if and only if the Galois action on its pro-ℓ fundamental group is unramified. This generalises the Neron–Ogg–Shafarevich criterion for abelian varieties, and was generalised further by Asada–Matumoto–Oda, who showed that even in the bad reduction case, several graph-theoretic invariants of the reduction type are determined by the Galois action.

In this talk, I will report on work with Netan Dogra, in which we examined the same problem for $\mathbb{Q}_\ell$-unipotent torsors of paths. The headline result is that there is a Galois-invariant $\mathbb{Q}_\ell$-unipotent path between two points if and only if they reduce onto the same component of the special fibre of a potential semistable model.

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Explicit methods in Chabauty–Kim theory for rational points

.... *Sachi Hashimoto :: Brown University, USA*

The Chabauty–Kim theory is a general framework to study rational points on curves through the study of quotients of fundamental groups. This theory has been made explicit in specific cases, yielding powerful computational tools for determining these rational point sets, such as the method of quadratic Chabauty.

I will present an overview of the current algorithmic challenges that appear when applying the quadratic Chabauty method to curves. I will discuss recent work to overcome some limitations of the method by computing p-adic local height functions on hyperelliptic curves at odd primes not equal to p. I will highlight examples and applications to Diophantine problems.

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Motivic aspects in genus zero

.... *Ishai Dan-Cohen :: Ben-Gurion University, Israel*

I will discuss joint work with S. Wewers and with D. Corwin from the previous decade concerning Chabauty-Kim theory for the thrice punctured line $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The Tannakian Galois group $G(Z)$ of the category of mixed Tate motives over an open subscheme Z of Spec $\mathbb{Z}$ decomposes as a semidirect product $G(Z) = U(Z) \rtimes \mathbb{G}_m$ with $U(Z)$ prounipotent. The p-adic period map associated to any p in Z gives rise to a highly nontrivial $\mathbb{Q}_p$-valued point u of $U(Z)$. The action of $U(Z)$ on the unipotent fundamental groupoid of X (possibly with additional punctures) gives rise to interesting functions on $U(Z)$ known as *motivic iterated integrals*; their values at u are special values of p-adic iterated integrals.

An intricate interplay between motivic and p-adic iterated integrals leads to effective methods for constructing locally analytic functions on $X(\mathbb{Z}_p)$ which vanish on $X(Z)$.

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